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Mass Term Variations in the Dirac Hydrogen Atom

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Abstract

The extended quaternion algebra allows *two* forms for the Dirac equation mass term. We show that the definition of angular momentum in the relativistic hydrogen atom suggests that either is physically allowed, but not a combination of both. A combination of both mass terms *could* still give the usual mass term in the nonrelativistic limit as is shown.

1. *Introduction*

We have recently shown that rest mass in nature causes the natural number system, complex quaternions, to be generalized from $\{e_\mu, ie_\mu\}$, $\mu = 0, 1, 2, 3$, to $\{e_\mu, ie_\mu, f_\mu, if_\mu\}$. This also naturally suggests the generalization of Lorentz symmetry from a 6 parameter group to a 10 parameter group. The reader is referred to our earlier papers for notation and details (Edmonds, 1974). Unless the Lorentz group is extended, there appear to be two equivalent extensions of the Weyl equation, $P\psi_a = 0$, to the Dirac equation, $P\psi_a = m(?)\psi_a$. In fact a *combination* of these extensions is Lorentz covariant and so it becomes ambiguous which should be associated with rest mass, or even if both might contribute somehow. If one were dominate, the other could be treated as a perturbation and would affect the hydrogen energy levels, in principle. We, therefore, look at this possibility.

2. *Dual Mass in the Dirac Equation*

The natural hypercomplex number system for relativistic physics with rest mass, $\{e_\mu, ie_\mu, f_\mu, if_\mu\}$, along with the Lorentz group, $\{LL^{\wedge} = 1e_0, L^{\vee} = L^{\wedge} = L_e\}$, allows us to write two mass terms in the Dirac equation

$$[P^\mu(e_\mu) - m_1(if_0) - m_2(f_0)]\,\psi_a = 0 \tag{2.1}$$

Under a Lorentz transformation, $P \to P' = L^\dagger P L$; $(if_0) \to (if_0)' = L^\dagger (if_0) L = L^\dagger L^{\dagger\wedge}(if_0)$, when $L^\vee = L^\wedge$; and $(f_0) \to (f_0)' = L^\dagger (f_0) L = (f_0) L^\wedge L = (f_0)$, for any L. Therefore, the m_1 and m_2 terms are both Lorentz invariant, when $[\;] \to L^\dagger [\;] L$, $\psi_a \to \psi_a' = L^\wedge \psi_a$, in equation (2.1).

We obtain the Klein-Gordon equation by forming

$$[P^\wedge + m_1(if_0) + m_2(f_0)]\,[P - m_1(if_0) - m_2(f_0)]\,\psi_a = 0 \qquad (2.2)$$

which gives, after expansion,

$$[P^\wedge P - (m_1{}^2 + m_2{}^2)(e_0)]\,\psi_a = 0,\; M^2 \equiv m_1{}^2 + m_2{}^2,\; P^\wedge P = P^\mu P_\mu(e_0) \quad (2.3)$$

The nonrelativistic limit is obtained from equation (2.3), so only M would appear in such low speed quantum physics! We, therefore, should turn to the relativistic hydrogen atom to see what effects m_1 and m_2 have on the energy spectrum. (The usual electron rest mass, measured in classical nonrelativistic e/m experiments is actually e/M.)

We note first that a free particle at rest has the equation

$$[P^0 - m_1(if_0) - m_2(f_0)]\,\psi_a = 0 \qquad (2.4)$$

Since (if_0) and (f_0) anticommute, though both commute with spin, (ie_3), we cannot chose ψ_a to be an eigenstate of both. Note that either would be the particle/antiparticle operator: $(f_0)(f_0) = (e_0)$, $(if_0)(if_0) = (e_0)$, $(f_0)\psi_a = +1\,\psi_a \Rightarrow P^0\psi_a = +m_2\psi_a$; $(f_0)\psi_a = -1\,\psi_a \Rightarrow P^0\psi_a = -m_2\psi_a$, etc. This would seem to indicate that m_1 or m_2 is not physical, but which? For the free particle we can show that they are basically equivalent. However, dropping the Lorentz restriction $L^\wedge = L^\vee$, puts (if_0) and (f_0) in very different perspectives. Now (f_0) is still invariant under $LL^\wedge = 1e_0$ but (if_0) becomes the fourth space dimension: $P = P^\mu(e_\mu) + P^4(if_0)$. We have previously speculated (Edmonds, 1975a) that this means that $P^4\psi_a = -m\psi_a$, so that the observed rest mass in three-space is associated with partons moving in four-space. We also considered (Edmonds, 1975b) $P^5\psi_a = -m\psi_a$, where $P^5(f_0)$ is, under $LL^\wedge = 1e_0$, a frame invariant coordinate. The natural candidate for this is cosmic time. The background radiation selects a natural rest frame (the isotropy frame). Locally this is preferred over all moving Lorentz frames and so its time coordinate is singled out as special. (Only the 15 parameter Conformal group, $LL^{-1} = 1e_0$ (determinant of $L \neq 0$), mixes (f_0) with the other five coordinates.)

Associating mass with cosmic time (P^5) and *also* considering P^4 as a physical but subnuclear (cyclic) coordinate, opens some interesting questions for *CP* and *T* symmetry. These are presently disturbing issues in physics, so this line of attack may be useful.

3. *Mass Term Selection and Angular Momentum*

The above discussion indicates that $m_1 = 0$ or $m_2 = 0$ in the Dirac equation, restricted to 3-space. We now consider each possibility in the hydrogen atom.

The Dirac equation in Hamiltonian form gives

$$[P^\mu(e_\mu) - \epsilon A^\mu(e_\mu) - m_1(if_0) - m_2(f_0)]\,\psi_a = 0 \qquad (3.1)$$

or

$$P^0(e_0)\psi_a = [-P^k(e_k) + \epsilon A^\mu(e_\mu) + m_1(if_0) + m_2(f_0)]\,\psi_a \equiv H\psi_a$$

The Coulomb potential gives $\epsilon A^0 = V(r), A^k = 0$. By analogy with basis vectors, we define "spherical" hypercomplex numbers

$$e_r \equiv \frac{x^k e_k}{r}, \quad e_\theta \equiv \frac{x^1 x^3 e_1 - x^2 x^3 e_2 - [(x^1)^2 + (x^2)^2] e_3}{r[(x^1)^2 + (x^2)^2]^{1/2}}$$

$$e_\phi \equiv \frac{-x^2 e_1 + x^1 e_2}{[(x^1)^2 + (x^2)^2]^{1/2}}, \quad r \equiv [-x^k x_k]^{1/2}, \quad e_r e_\theta = (ie_\phi) = -e_\theta e_r$$

$$e_\theta e_\phi = (ie_r) = -e_\phi e_\theta, \quad e_\phi e_r = (ie_\theta) = -e_r e_\phi, \quad e_r{}^\wedge = -e_r, \text{ etc.} \tag{3.2}$$

The standard trick to getting H in "spherical" form is to note that

$$P^k e_k \equiv \mathbf{P} = \left(\frac{-\mathbf{x}\mathbf{x}^\wedge}{r^2}\right)\mathbf{P} = \frac{-\mathbf{x}}{r^2}\,[\tfrac{1}{2}(\mathbf{x}^\wedge\mathbf{P} + \mathbf{P}^\wedge\mathbf{x}) + \tfrac{1}{2}(\mathbf{x}^\wedge\mathbf{P} - \mathbf{P}^\wedge\mathbf{x})]$$

$$= \frac{-e_r}{r}\,[\tfrac{1}{2}(\mathbf{x}^\wedge\mathbf{P} + \mathbf{P}^\wedge\mathbf{x}) - i\left(\frac{i}{2}\right)(\mathbf{x}^\wedge\mathbf{P} - \mathbf{P}^\wedge\mathbf{x})] \tag{3.3}$$

For reasons related to the nonrelativistic limit, we define

$$J \equiv \frac{i}{2}\,[\mathbf{x}^\wedge\mathbf{P} - \mathbf{P}^\wedge\mathbf{x}] \equiv -iJ^k(ie_k), \quad J^1 = (x^2P^3 - x^3P^2)(e_0) - i\,\frac{\hbar}{2}(ie_1) \tag{3.4}$$

But J does not commute with H, so one defines

$$K \equiv J - \frac{\hbar}{2}(e_0) = K^{\wedge\dagger} = K^0(e_0) + K^j(ie_j) \tag{3.5}$$

where $-\hbar/2$ is *chosen* so that K anticommutes with e_r. Returning to $P^k e_k$ we can now write

$$P^k e_k = -\frac{e_r}{r}\left[\frac{3}{2}i\hbar + (x^kP_k) - iJ\right] = -\frac{e_r}{r}\left[\frac{3}{2}i\hbar + i\hbar r\frac{\partial}{\partial r} - iJ\right]$$

$$= -\frac{e_r}{r}\left[\frac{3}{2}i\hbar + i\hbar r\frac{\partial}{\partial r} - i\left(K + \frac{\hbar}{2}\right)\right] = e_r\left[-i\hbar\frac{\partial}{\partial r} - i\frac{\hbar}{r} + i\frac{K}{r}\right] \tag{3.6}$$

It can be shown that K commutes with $\partial/\partial r$ and $1/r$, but anticommutes with e_r (and this is why J is not useful in relativistic physics). Now notice the mass terms, (if_0) and (f_0). Because $K = K^{\wedge\dagger}$, it commutes with (f_0). Because it also has no (f) parts, $K^\wedge = K^\vee$, it commutes with (if_0). Since e_r is made of $\{e_k\}$, it anticommutes with (f_0) and (if_0). Therefore, either $(f_0)K$ or $(if_0)K$ will serve as the angular momentum operator. No preference for m_1 or m_2 has emerged. We do see that both $m_1(if_0)$ and $m_2(f_0)$ cannot appear in the hydrogen atom without disrupting the angular momentum operator commutation with H. This is because (f_0) and (if_0) anticommute.

Notice that either $[(f_0)K]$ or $[(if_0)K]$, of the form $\{f_0, if_k\}$ or $\{if_0, f_k\}$, is the *physical* angular momentum operator, not K or J, which both involve $\{e_0, ie_k\}$. We next look at the internal hypercomplex structure of $(f_0)K$ and $(if_0)K$. We can use $P^j x^k = x^k P^j - i\hbar\delta^{kj}$ to rewrite K in the form

$$K = \hbar e_0 - \frac{i}{2}[(\mathbf{xP}) - (\ )^\dagger], \quad \mathbf{x}^\dagger = \mathbf{x}, \mathbf{P}^\dagger = \mathbf{P}, \quad (\mathbf{xP})^\dagger = x^k P^j e_j e^k \quad (3.7)$$

We now see that $(if_0)K$ is going to give $(if_0)\mathbf{xP} = x^k(if_k)P^j e_j$, whereas $(f_0)K$ is going to give $(f_0)\mathbf{xP} = x^k(f_k)P^j e_j$. Therefore, if we write the angular momentum operator in compact hypercomplex number form, we see that $\mathbf{x} = x^k(f_k)$ or $x^k(if_k)$, depending upon the choice of $m(f_0)$ or $m(if_0)$.

The angular momentum operator shows that P and x should be written in *different* hypercomplex forms, due to rest mass. The natural question to ask next is whether $x^k(f_k)$ or $x^k(if_k)$ can be distinguished by their Lorentz properties. We notice that

$$x' = L^\wedge x L, \quad x \equiv +x^\wedge \Rightarrow x'^\wedge = x', \quad x \leftrightarrow \{f_\mu, ie_0, e_0\} \quad (3.8)$$

and $L^\wedge(e_0)L = (e_0)L^\wedge L = (e_0)$. Because $L^\wedge = L^\vee = L_e$, we also have $L^\wedge(ie_0)L = (ie_0)L^\wedge L = (ie_0)$. Thus $\{f_\mu\}$ represents a 4-vector. This would indicate a preference for $x^\mu(f_\mu)$ and hence $m_2(f_0)$. However, the third conjugation $(\ )^\vee$, of the set $(\ )^\wedge, (\ )^\dagger, (\ )^\vee, (\ )^{\vee\dagger *}$, can be used to write

$$x' = L^\vee x L, \quad x = +x^\vee \Rightarrow x'^\vee = x', \quad x \leftrightarrow \{if_\mu, e_0, ie_0\} \quad (3.9)$$

and $L^\vee(ie_0)L = (ie_0)L^\wedge L = (ie_0)$. Because $L^\wedge = L^\vee$, we also have $L^\vee(e_0)L = (e_0)L^\vee L = (e_0)L^\wedge L = (e_0)$. Thus $\{if_\mu\}$ also represents a 4-vector!

4. *Conclusion*

The first conclusion we can draw is that Lorentz symmetry $LL^\wedge = 1e_0$, $L^\wedge = L^\vee$, has two natural generalizations, $LL^\wedge = 1e_0$ and $LL^\vee = 1e_0$. These are isomorphic, though we have chosen $LL^\vee = 1e_0$. As a result $\{e_\mu\}$ picks up (if_0) to form the 5-vectors. If we had chosen $LL^\vee = 1e_0$, then $\{e_\mu\}$ would pick up (f_0) to form the 5-vectors. Thus there is no way to choose between $m_1(if_0)$ and $m_2(f_0)$. If nature is *only* Lorentz invariant, then particles in nature can exist with either kind of invariant mass. This could be an important distinction, e.g., hadrons and leptons or such!

The physical angular momentum operators which commute with H are $[(f_0)K]$ and $[(f_0)K]^2$, with the eigenvalues $k = \pm 1, \pm 2, \ldots$ and $k^2 = 1, 4, 9, \ldots$, respectively. The third operator that commutes with these is J^3, with eigenvalues $\{1/2, -1/2\}$, $\{3/2, 1/2, -1/2, -3/2\}, \ldots = \{|k| - 1/2,$

$\ldots -(|k| - 1/2)\}$. It is only here that half integers appear. This operator has no elegant form in the hypercomplex number language. In the past J was considered the physical operator but $[(f_0)K]$ is used in the relativistic equation and, therefore, should be useful in the nonrelativistic case.

We have recently considered the gravity equation using this hypercomplex number formalism. It appears that 4-vector and 5-vector curvature equations can be readily accommodated in this language but not 6-vectors. Therefore, if we choose $\{e_\mu, if_0\}$ for curved 5-space, then m_1 and m_2 would have quite different relations to the full theory, since (if_0) becomes position dependent but (f_0) remains independent of position. This may indicate that both mass types are important and physically quite *distinct* for curved space-time quantum theory.

Finally, we note that (f_0) does not commute with H and (f_0) is the particle/antiparticle operator, as can be seen in the free particle equation (2.4). Therefore, the hydrogen atom eigenstates are not particle/antiparticle eigenstates!

References

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